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# Matrix elements for the symplectic $sp(4)$ Lie algebra

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**Abstract.** The closed Gelfand–Tsetlin type formulae are constructed in this paper for the matrix elements of the symplectic $sp(4)$ Lie algebra in an arbitrary irreducible representation using the reduction chain $sp(4) \supset sp(2) \oplus sp(2)$. All resulting formulae for the matrix elements were obtained by directly solving the Cartan–Weyl commutation relations of the $sp(4)$ algebra.

## 1. Introduction

The explicit construction of irreducible matrices is generally a difficult practical problem in the representation theory of Lie algebras. A prototype to this problem is the well known construction of matrix elements for the angular momentum $su(2)$ algebra [1]. The generalization for the unitary and orthogonal algebras was given by Gelfand and Tsetlin [2–4]. The Gelfand–Tsetlin method provides suitable quantum numbers to label the vectors of the bases corresponding to the canonical reductions $su(n) \supset su(n-1)$ and $so(n) \supset so(n-1)$. It also provides closed formulae for the matrix elements in terms of the given quantum numbers. Unfortunately, the same program has not yet been developed for the symplectic algebras [4–8].

In this work, we construct the Gelfand–Tsetlin-type formulae for the matrix elements of the symplectic $sp(4)$ Lie algebra in an arbitrary irreducible representation using the canonical reduction chain $sp(4) \supset sp(2) \oplus sp(2)$. The commutation relations for the generators associated with the simple roots were solved by reducing them to recursion relations. Similar formulae for the remaining elements were obtained by taking successive commutators of the simple generators. The present result is intended to be helpful in applications in which the handling of algebraic matrix elements in several different irreducible representations of the symplectic $sp(4)$ Lie algebra can be used.

Although the local isomorphism $sp(4) \sim so(5)$ is well known, there are some reasons to develop the Gelfand–Tsetlin formulae for the $sp(4)$ algebra itself. The first reason is that the isomorphism is local and not global. This means that the present matrix elements can be useful in calculating the matrices of the corresponding group elements. The second reason is that all quantum numbers used here are explicitly given by the branching rules of the chain $sp(4) \supset sp(2) \oplus sp(2)$ without the intermediate $so(5)$ quantum numbers. Therefore, the matrix elements presented here can be used in previous applications using the $so(5)$ algebra [9–11]. Another reason is that the Gelfand–Tsetlin formulae for the $so(5)$ algebra do not give irreducible matrices whose commutation relations are directly in the canonical Cartan–Weyl form [4] whereas the irreducible matrices given here do. This feature can be helpful in obtaining the analogous deformed $sp(4)$ formulae to the unitary deformed algebras

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(or 'quantum groups') [12–14]. Finally, the present technique can be helpful in the achievement of the corresponding matrix elements of the $sp(6)$ algebra which has important applications in nuclear physics [15] and, more recently, in molecular genetics [16, 17].

This paper is organized in four sections. The matrix elements are presented in section 2. The method followed in the calculation of the matrix elements is described in section 3; and section 4 is reserved for summary and conclusions.

## 2. Matrix elements for the symplectic $sp(4)$ algebra

Section 2.1 introduces the basic notation for roots and weights as well the canonical form for the commutation relations; and section 2.2 presents the branching rules for the chain $C_2 \supset C_1 \oplus C_1$. The matrix elements and their properties are presented in section 2.3. The correspondence with previous works using the isomorphism $sp(4) \sim so(5)$ and two special cases are discussed in section 2.4.

### *2.1. The classical $C_2$ algebra*

The classical semisimple $C_2 \sim sp(4)$ Lie algebra can be defined by the Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}. \tag{1}$$

Once the Cartan matrix is known, the corresponding simple roots can be written in three different bases as

$$\alpha_1 = \{1, 0\} = (2, -1) = [1, -1] \tag{2}$$

$$\alpha_2 = \{0, 1\} = (-2, 2) = [0, 2] \tag{3}$$

where the longest root was normalized to two and the notation is from Chen [18]. A pair of braces, { }, denotes the basis formed by the simple roots (SRS), while a pair of parentheses ( ), denotes the basis formed by the weights of the basic irreducible representations (Dynkin labels, DYN). Square brackets, [ ], denote the basis formed by the positive weights of the fundamental irreducible representation (Cartan or cartesian labels, FWS). Only the FWS basis is orthonormal and, therefore, it is useful to calculate scalar products among roots or weights. The last two positive roots

$$\alpha_3 = \{1, 1\} = (0, 1) = [1, 1] \tag{4}$$

$$\alpha_4 = \{2, 1\} = (2, 0) = [2, 0] \tag{5}$$

complete the whole set of positive roots. As usual, each element in the algebra can be put in correspondence with a root:

$$H_i \to (0, 0) \tag{6}$$

$$E_i^+ \to \alpha_i \qquad i = 1, 2. \tag{7}$$

The elements $\{H_1, H_2, E_1^+, E_2^+\}$ are the generators of the $sp(4)$ algebra and $\{H_1, H_2\}$ are the generators of the Abelian Cartan subalgebra. There are two other elements, $E_3^+$ and $E_4^+$, associated with the positive roots $\alpha_3$ and $\alpha_4$, respectively, which can be defined as

$$E_3^+ \equiv \frac{1}{\sqrt{2}}[E_1^+, E_2^+] \tag{8}$$

$$E_4^+ \equiv \frac{1}{\sqrt{2}}[E_1^+, E_3^+]. \tag{9}$$

All four elements $E_i^+$ are related to the elements $E_i^-$ associated with the negative roots through

$$E_i^- = (E_i^+)^\dagger. \tag{10}$$

In this way, the defining commutation relations satisfied by those elements are

$$[H_i, H_j] = 0 \qquad i, j = 1, 2 \tag{11}$$

$$[H_i, E_j^\pm] = \pm(\alpha_j)_i E_j^\pm \qquad i = 1 \ldots 2 \qquad j = 1 \ldots 4 \tag{12}$$

$$[E_i^+, E_i^-] = (\alpha_i)_1 H_1 + (\alpha_i)_2 H_2 \qquad i = 1 \ldots 4 \tag{13}$$

where the simple roots must be written in the basis formed by the positive weights of the fundamental irreducible representation (FWS). We can see from the defining relations (11)–(13) that the $sp(4)$ algebra has two $C_1 \sim sp(2)$ subalgebras: one formed by $H_2$ and $E_2^\pm$ and another formed by $H_1$ and $E_4^\pm$.

It can be useful, for future applications involving deformed algebras, to give the corresponding Chevalley form of the $sp(4)$ algebra. After the simple identification

$$h_1 \equiv H_1 - H_2 \qquad e_1 \equiv E_1^+ \qquad f_1 \equiv E_1^- \tag{14}$$

$$h_2 \equiv 2H_2 \qquad e_2 \equiv \frac{1}{\sqrt{2}} E_2^+ \qquad f_2 \equiv \frac{1}{\sqrt{2}} E_2^- \tag{15}$$

the Chevalley form is obtained. Now the commutation relations are

$$[h_i, h_j] = 0 \tag{16}$$

$$[h_i, e_j] = +a_{ij} e_j \tag{17}$$

$$[h_i, f_j] = -a_{ij} f_j \tag{18}$$

$$[e_i, f_j] = \delta_{i,j} h_i \qquad i, j = 1, 2 \tag{19}$$

where $a_{ij}$ are the matrix elements of the Cartan matrix (1). The Serre relations can be written as

$$\sum_{k=0}^{2} (-1)^k \binom{1 - a_{ij}}{k} e_i^k e_j e_i^{1-a_{ij}-k} = 0 \qquad i \neq j = 1, 2. \tag{20}$$

When the generators $e$ are replaced by the corresponding generators $f$, then another set of Serre relations is obtained. Note that these relations are also satisfied by the generators in the Cartan–Weyl form.

### *2.2. Branching rule for the chain $C_2 \supset C_1 \oplus C_1$*

One important step in obtaining the matrix elements themselves is the construction of the complete set of commuting operators (CSCO). The CSCO are invariant operators formed by the elements of the given algebra and by the elements of its subalgebras. In general it is possible to have many different reduction chains of subalgebras. A particular chain can be chosen regarding the intended application. The eigenvalues of the CSCO (quantum numbers) give enough information to label each vector (eigenfunction) of any irreducible representation (IRREP) in a unique form. A difficult task is to discover how a given IRREP contains the IRREPs of the subalgebras in a chosen reduction chain. The solution to this problem is better known as branching rules. In another words, given an IRREP characterized by a given highest weight, the branching rules tell us how to compute, from the given highest weight, all the other highest weights (and its weight systems) for the IRREPs of the subalgebras. The Gelfand–Tsetlin method [4], e.g., gives a very efficient prescription to accomplish this for the unitary and orthogonal algebras.

**Table 1.** The weight system in the FWS base for the first three low-dimensional IRREPs of $sp(4)$ algebra.

| | [1, 0] | | | | | [1, 1] | | | | | [2, 0] | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\sigma_1$ | $\sigma_2$ | $\gamma_{12}$ | $h_1$ | $h_2$ | $\sigma_1$ | $\sigma_2$ | $\gamma_{12}$ | $h_1$ | $h_2$ | $\sigma_1$ | $\sigma_2$ | $\gamma_{12}$ | $h_1$ | $h_2$ |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 0 | 2 | 2 | 0 |
| 2 | 1 | 0 | 1 | −1 | 0 | 1 | 1 | 1 | −1 | −1 | 2 | 0 | 2 | −2 | 0 |
| 3 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | −1 | 1 | 1 | 1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 |
| 5 | | | | | | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | −1 |
| 6 | | | | | | | | | | | 1 | 1 | 1 | −1 | 1 |
| 7 | | | | | | | | | | | 0 | 2 | 0 | 0 | 2 |
| 8 | | | | | | | | | | | 0 | 2 | 0 | 0 | −2 |
| 9 | | | | | | | | | | | 0 | 2 | 0 | 0 | 0 |
| 10 | | | | | | | | | | | 2 | 0 | 2 | 0 | 0 |

Although a lot of generating functions have been developed to give the branching rules of many chains of Lie algebras [19], only recently was the branching rule for the reduction $C_n \supset C_{n-1} \oplus C_1$ analytically solved [20]. For the $sp(4)$ algebra, each vector in an irreducible representation given by the highest weight

$$\omega_2 = [\omega_{12}, \omega_{22}] = (\omega_{12} - \omega_{22}, \omega_{22}) = \{\omega_{12}, \tfrac{1}{2}(\omega_{12} - \omega_{22})\} \qquad (21)$$

is labelled by the quantum numbers (in the FWS basis) of the $sp(4) \supset sp(2) \oplus sp(2)$ canonical chain as follows:

$$\left| \begin{array}{cccc} \omega_{12} & & \omega_{22} & \\ & \gamma_{12} & & h_2 \\ & & \omega_{11} & \\ & & & h_1 \end{array} \right\rangle \equiv |\sigma_1, \sigma_2(\gamma_{12}), h_1, h_2\rangle \qquad \begin{array}{l} \omega_{12} \geqslant \gamma_{12} \geqslant \omega_{22}, \gamma_{12} \geqslant \omega_{11} \geqslant \gamma_{12} - \omega_{22} \\ \sigma_1 = \omega_{11} \quad \sigma_2 = \omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12} \\ h_i = \sigma_i \quad \sigma_i - 2, \ldots, -\sigma_i \quad i = 1, 2. \end{array} \qquad (22)$$

The dimension of each IRREP is

$$\dim([\omega_{12}, \omega_{22}]) = \tfrac{1}{6}(2 + \omega_{11})(\omega_{12} + 1)(1 + \omega_{11} - \omega_{12})(3 + \omega_{11} + \omega_{12}). \qquad (23)$$

The quantum numbers $\sigma_i$ are the highest weights of the $sp(2) \oplus sp(2)$ irreducible representations and $[h_1, h_2]$ are the weights composing the weight system of the highest weight $\omega_2$. The reduction chain $sp(4) \supset sp(2) \oplus sp(2)$ is canonical: it means that the IRREPs of the $sp(2)$ subalgebras appear only once in a given IRREP of the $sp(4)$ algebra. This is not the case for the higher-dimensional symplectic algebras [19, 20]. It is worth considering the first three IRREPs of the $sp(4)$ explicitly. The first two low-dimensional IRREPs are the two basic IRREPs: the fundamental IRREP of dimension four given by the highest weight $[1, 0] = (1, 0)$ and the basic IRREP of dimension five given by the highest weight $[1, 1] = (0, 1)$. The adjoint IRREP of dimension ten given by $[2, 0] = (2, 0)$ is the third low-dimensional one. The corresponding weights are shown in table 1. The next section describes the actions for all elements of the $sp(4)$ algebra on the basis (22).

### *2.3. The matrix elements*

This section presents closed formulae to calculate explicitly the matrix elements of the generators of the $sp(4)$ algebra in terms of the quantum numbers (22). The corresponding matrices satisfy the relation (10) and the commutation relations in the Cartan–Weyl form (11)–(13).

**Table 2.** Relative phase in front of the coefficients of the actions of $E_1^{\pm}$ and $E_3^{\pm}$ elements.

| $A^{\pm}$ | $B^{\pm}$ | $C^{\pm}$ | $D^{\pm}$ | $A'^{\pm}$ | $B'^{\pm}$ | $C'^{\pm}$ | $D'^{\pm}$ |
|---|---|---|---|---|---|---|---|
| + | $\pm$ | $\mp$ | + | $\mp$ | + | + | $\pm$ |
| + | $\mp$ | $\pm$ | + | $\mp$ | − | − | $\pm$ |
| $\pm$ | + | + | $\mp$ | − | $\pm$ | $\mp$ | − |
| $\mp$ | + | + | $\pm$ | + | $\pm$ | $\mp$ | + |

Some matrix elements have very simple formulae. For example, since in the Cartan–Weyl form the weights are the eigenvalues of the generators in the Cartan subalgebra, then, by definition,

$$H_i|\sigma_1, \sigma_2, h_1, h_2\rangle = h_i|\sigma_1, \sigma_2, h_1, h_2\rangle \qquad i = 1, 2. \tag{24}$$

Another set of very simple actions are those regarding the raising (lowering) operators $E_i^{\pm}$, $i = 2, 4$, of the $sp(2)$ algebras. These actions are similar to the action of raising (lowering) angular momentum operators of the $su(2) \sim sp(2)$ algebra and, therefore, they can be constructed easily (see the appendix for a simple derivation):

$$E_2^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = \{\tfrac{1}{2}(\sigma_2 \mp h_2)(\sigma_2 \pm h_2 + 2)\}^{\frac{1}{2}}|\sigma_1, \sigma_2, h_1, h_2 \pm 2\rangle \tag{25}$$

$$E_4^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = \{\tfrac{1}{2}(\sigma_1 \mp h_1)(\sigma_1 \pm h_1 + 2)\}^{\frac{1}{2}}|\sigma_1, \sigma_2, h_1 \pm 2, h_2\rangle. \tag{26}$$

These operators do not couple irreducible representations of $sp(2)$ subalgebras because they are pure $sp(2)$ elements inside the $sp(4)$ algebra.

Unlike the previous ones, the $E_1^{\pm}$ generators, which are genuine $sp(4)$ elements, couple irreducible representations of the $sp(2)$ subalgebras (see section 3 for a deduction):

$$\begin{aligned} E_1^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = {} & A^{\pm}|\sigma_1 + 1, \sigma_2 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & \pm B^{\pm}|\sigma_1 + 1, \sigma_2 - 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & \mp C^{\pm}|\sigma_1 - 1, \sigma_2 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & + D^{\pm}|\sigma_1 - 1, \sigma_2 - 1, h_1 \pm 1, h_2 \mp 1\rangle. \end{aligned} \tag{27}$$

The remaining $sp(4)$ elements, $E_3^{\pm}$, defined by the first commutation relation in (8), have the following matrix elements:

$$\begin{aligned} E_3^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = {} & \mp A'^{\pm}|\sigma_1 + 1, \sigma_2 + 1, h_1 \pm 1, h_2 \pm 1\rangle \\ & + B'^{\pm}|\sigma_1 + 1, \sigma_2 - 1, h_1 \pm 1, h_2 \pm 1\rangle \\ & + C'^{\pm}|\sigma_1 - 1, \sigma_2 + 1, h_1 \pm 1, h_2 \pm 1\rangle \\ & \pm D'^{\pm}|\sigma_1 - 1, \sigma_2 - 1, h_1 \pm 1, h_2 \pm 1\rangle \end{aligned} \tag{28}$$

where the primed coefficients are obtained from the unprimed ones just by exchanging the sign of $h_2$:

$$X'^{\pm} = X^{\pm}(h_2 \to -h_2) \qquad X \in \{A, B, C, D\}. \tag{29}$$

Note that only the first neighbourhood ($\sigma_i \pm 1$) irreducible representations of the $sp(2)$ algebras are coupled by $E_1^{\pm}$ and $E_3^{\pm}$ elements of the $sp(4)$ algebra. The relative phases in front of the $A^{\pm}$, $B^{\pm}$, $C^{\pm}$ and, $D^{\pm}$ coefficients in equation (27) can be chosen in four different ways as shown in table 2. All of them are consistent with the Cartan–Weyl commutation relations (see section 3).

The coefficients $A^+$, $B^+$, $C^+$ and $D^+$ depend on the quantum numbers (22) as follows:

$$A^+ = \left\{ \frac{(\omega_+ - \sigma_+)(\omega_+ + \sigma_+ + 6)(\sigma_+ - \omega_- + 2)(\sigma_+ + \omega_- + 4)}{64(\sigma_1 + 1)(\sigma_1 + 2)(\sigma_2 + 1)(\sigma_2 + 2)} \right\}^{\frac{1}{2}} \times \{(\sigma_1 + h_1 + 2)(\sigma_2 - h_2 + 2)\}^{\frac{1}{2}} \tag{30}$$

$$D^+ = \left\{ \frac{(\omega_+ - \sigma_+ + 2)(\omega_+ + \sigma_+ + 4)(\sigma_+ - \omega_-)(\sigma_+ + \omega_- + 2)}{64\sigma_1(\sigma_1 + 1)\sigma_2(\sigma_2 + 1)} \right\}^{\frac{1}{2}} \times \{(\sigma_1 - h_1)(\sigma_2 + h_2)\}^{\frac{1}{2}} \tag{31}$$

$$B^+ = \left\{ \frac{(\omega_- + \sigma_- + 2)(\omega_- - \sigma_-)(\omega_+ - \sigma_- + 2)(\omega_+ + \sigma_- + 4)}{64(\sigma_1 + 1)(\sigma_1 + 2)\sigma_2(\sigma_2 + 1)} \right\}^{\frac{1}{2}} \times \{(\sigma_1 + h_1 + 2)(\sigma_2 + h_2)\}^{\frac{1}{2}} \tag{32}$$

$$C^+ = \left\{ \frac{(\omega_- + \sigma_-)(\omega_- - \sigma_- + 2)(\omega_+ - \sigma_- + 4)(\omega_+ + \sigma_- + 2)}{64\sigma_1(\sigma_1 + 1)(\sigma_2 + 1)(\sigma_2 + 2)} \right\}^{\frac{1}{2}} \times \{(\sigma_1 - h_1)(\sigma_2 - h_2 + 2)\}^{\frac{1}{2}} \tag{33}$$

where

$$\omega_\pm = \omega_{12} \pm \omega_{22} \qquad \sigma_\pm = \sigma_1 \pm \sigma_2. \tag{34}$$

All coefficients (30)–(33) are real numbers and the sum $\sigma_+$ is only present in $A^+$ and $D^+$ while the difference $\sigma_-$ is only present in $B^+$ and $C^+$. In addition to this, the $D^+$ and $C^+$ coefficients can be obtained from $A^+$ and $B^+$ in the following manner:

$$D^+ = A^+(\sigma_1 \to \sigma_1 - 1, \sigma_2 \to \sigma_2 - 1, h_1 \to -(h_1 + 1), h_2 \to -(h_2 - 1)) \tag{35}$$

$$C^+ = B^+(\sigma_1 \to \sigma_1 - 1, \sigma_2 \to \sigma_2 + 1, h_1 \to -(h_1 + 1), h_2 \to -(h_2 - 1)). \tag{36}$$

As a consequence of (10), the corresponding lowering coefficients are related to the raising ones as follows:

$$\begin{aligned} A^-(\sigma_1, \sigma_2, h_1, h_2) &= D^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \\ B^-(\sigma_1, \sigma_2, h_1, h_2) &= C^+(\sigma_1 + 1, \sigma_2 - 1, h_1 - 1, h_2 + 1) \\ C^-(\sigma_1, \sigma_2, h_1, h_2) &= B^+(\sigma_1 - 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \\ D^-(\sigma_1, \sigma_2, h_1, h_2) &= A^+(\sigma_1 - 1, \sigma_2 - 1, h_1 - 1, h_2 + 1). \end{aligned} \tag{37}$$

For completeness, the action of the Casimir operator, or the second-order invariant, of the $sp(4)$ algebra and as well those of the $sp(2)$ subalgebras,

$$K_2 = H_1^2 + H_2^2 + \sum_{i=1}^{4} [E_i^+, E_i^-]_+ \tag{38}$$

$$J_1 = H_1^2 + [E_4^+, E_4^-]_+ \tag{39}$$

$$J_2 = H_2^2 + [E_2^+, E_2^-]_+ \tag{40}$$

where $[a, b]_+ = ab + ba$, were also calculated to ensure that the representation is properly adapted to the used chain:

$$K_2|\omega_{12}, \omega_{22}, \sigma_1, \sigma_2, h_1, h_2\rangle = [\omega_{12}(\omega_{12} + 4) + \omega_{22}(\omega_{22} + 2)]|\omega_{12}, \omega_{22}, \sigma_1, \sigma_2, h_1, h_2\rangle \tag{41}$$

$$J_i|\omega_{12}, \omega_{22}, \sigma_1, \sigma_2, h_1, h_2\rangle = \sigma_i(\sigma_i + 2)|\omega_{12}, \omega_{22}, \sigma_1, \sigma_2, h_1, h_2\rangle \qquad i = 1, 2. \tag{42}$$

### *2.4. Special cases*

Let us consider here two special cases of irreducible representations: (I) the symmetric IRREPs given by $[\omega_{12}, 0] = (\omega_{12}, 0)$ and, (II) the anti-symmetric IRREPs given by $[\omega_{12}, \omega_{12}] = (0, \omega_{12})$. The respective dimensions are

$$\dim([\omega_{12}, 0]) = \tfrac{1}{6}(2 + \omega_{12})(\omega_{12} + 1)(3 + \omega_{12}) \tag{43}$$

$$\dim([\omega_{12}, \omega_{12}]) = \tfrac{1}{6}(2 + \omega_{12})(\omega_{12} + 1)(3 + 2\omega_{12}). \tag{44}$$

For the symmetric case, the branching rules (22) give $\sigma_1 = \gamma_{12}$ and $\sigma_2 = \omega_{12} - \sigma_1$. Therefore, from equations (34), we have $\sigma_+ - \omega_\pm = 0$, which means that only the non-null matrix elements of $E_1$ are given by the coefficients $B$ and $C$:

$$\begin{aligned} E_1^{\pm}|\sigma_1, h_1, h_2\rangle &= \pm\tfrac{1}{2}\{(\sigma_1 + h_1 + 2)(\sigma_2 + h_2)\}^{\frac{1}{2}}|\sigma_1 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ &\quad \mp\tfrac{1}{2}\{(\sigma_1 - h_1)(\sigma_2 - h_2 + 2)\}^{\frac{1}{2}}|\sigma_1 - 1, h_1 \pm 1, h_2 \mp 1\rangle. \end{aligned} \tag{45}$$

For the anti-symmetric case, the branching rules (22) give $\gamma_{12} = \omega_{12}$ and $\sigma_2 = \sigma_1$. Therefore, from equations (34), we have $\sigma_- \pm \omega_- = 0$, which means that only the non-null matrix elements of $E_1$ are now given by the coefficients $A$ and $D$:

$$\begin{aligned} E_1^{\pm}|\sigma_1, h_1, h_2\rangle &= \frac{1}{2}\left\{\frac{(\omega_{12} - \sigma_1)(\omega_{12} + \sigma_1 + 3)(\sigma_1 + h_1 + 2)(\sigma_1 - h_2 + 2)}{(\sigma_1 + 1)(\sigma_1 + 2)}\right\}^{\frac{1}{2}} \\ &\quad \times|\sigma_1 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ &\quad + \frac{1}{2}\left\{\frac{(\omega_{12} - \sigma_1 + 1)(\omega_{12} + \sigma_1 + 2)(\sigma_1 - h_1)(\sigma_1 + h_2)}{\sigma_1(\sigma_1 + 1)}\right\}^{\frac{1}{2}} \\ &\quad \times|\sigma_1 - 1, h_1 \pm 1, h_2 \mp 1\rangle. \end{aligned} \tag{46}$$

It is worth computing the matrices of the fundamental irreducible representation [1, 0]. The ordering of the vectors is shown in the first column of table 1. The matrices for this IRREP can be written in a convenient form using the $4 \times 4$ Weyl matrices $A_{ij}$ as shown in table 3. The Weyl matrices has only a unit nun-null element in the $ij$ entry and they obey the following commutation relation:

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{li}A_{kj}. \tag{47}$$

The correspondence between the parametrization used in the present work and that used by K T Hecht [9] (table 1, p 180) is also shown in table 3. Using this correspondence we can see that the matrix elements of $F_{-\frac{1}{2},-\frac{1}{2}}$ calculated by Hecht [9] (section 4, p 184) are precisely the matrix elements of $E_3^-$ given by equation (28) after the identification $\sigma_1 = 2\Lambda$, $h_1 = 2M_\Lambda$, $\sigma_2 = 2J$ and $h_2 = 2M_J$. It should be remembered here that the matrix elements calculated by Hecht [9] are the matrix elements of the elements of the orthogonal $so(5)$ algebra in the chain $su(4) \supset so(5) \supset so(4)$ and also the local isomorphisms $sp(4) \sim so(5)$, $so(4) \sim su(2) \oplus su(2)$ and $sp(2) \sim su(2)$.

## 3. Solution for the commutation relations

This section describes the method used to solve the commutation relations for the generators of the $sp(4)$ algebra in order to find their matrix elements. The fundamental ideia is to solve the commutation relations by transforming them in recurrence relations. The matrix elements for the generators of the $sp(2)$ subalgebras as well those of the Cartan subalgebra are calculated in section 3.1. The pure $sp(4)$ generator and its selection rules in the chain $sp(4) \supset sp(2) \oplus sp(2)$ are identified in section 3.2. Using the Cartan–Weyl commutation

**Table 3.** The matrices of the fundamental representation and the correspondence between the parametrizations of the $sp(4)$ algebra used in this work and by Hecht [9].

| This work | Hecht | This work | Hecht |
|---|---|---|---|
| $H_1 = A_{11} - A_{22}$ | $= 2H_1$ | $H_2 = A_{33} - A_{44}$ | $= 2H_2$ |
| $E_1^+ = A_{13} - A_{42}$ | $= \sqrt{2}F_{\frac{1}{2},-\frac{1}{2}}$ | $E_1^- = A_{31} - A_{24}$ | $= \sqrt{2}F_{-\frac{1}{2},\frac{1}{2}}$ |
| $E_2^+ = \sqrt{2}A_{34}$ | $= \frac{1}{\sqrt{2}}F_{0,1}$ | $E_2^- = \sqrt{2}A_{43}$ | $= \frac{1}{\sqrt{2}}F_{0,-1}$ |
| $E_3^+ = A_{14} + A_{32}$ | $= \sqrt{2}F_{\frac{1}{2},\frac{1}{2}}$ | $E_3^- = A_{41} + A_{23}$ | $= \sqrt{2}F_{-\frac{1}{2},-\frac{1}{2}}$ |
| $E_4^+ = \sqrt{2}A_{12}$ | $= \frac{1}{\sqrt{2}}F_{1,0}$ | $E_4^- = \sqrt{2}A_{21}$ | $= \frac{1}{\sqrt{2}}F_{-1,0}$ |

relations involving the raising–lowering pure $sp(4)$ generator, the corresponding recurrence relations are established in section 3.3 and solved in section 3.4.

### 3.1. Matrix elements for the subalgebras

Some formulae presented in subsection 2.3 are easily obtained. For example, since by definition the weights $h_i$ of an irreducible representation are the eigenvalues of the elements $H_i$ composing the Cartan subalgebra, then the equation (24) is established. Other simple matrix elements are those of the raising (lowering) generators of the $sp(2)$ subalgebras. As shown in the appendix, the commutation relations in this case are directly solved without any difficulty. These results are the same for the higher-dimensional symplectic algebras.

### 3.2. The selection rules for the pure $sp(4)$ generator

In general, the raising–lowering operators associated with the non-simple roots can be defined by the raising-lowering operators associated with the simple roots. For example, the elements $E_3$ and $E_4$ associated with the non-simple roots $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$ are defined by the generators $E_1$ and $E_2$, associated with the simple roots $\alpha_1$ and $\alpha_2$, as shown in equations (8) and (9). Thus, for the present case, we only have to solve one set of recurrence relations supplied by the commutation relation of the generator $E_1$.

Before we establish the recurrence relations, we must to know how the operator $E_1$ will act on a generic base vector given in (22). Naturally, the action proposed in equation (27) is the simplest one which is compatible with the branching rules (22) and the root $\alpha_1 = [1, -1]$. If it is assumed that the basis (22) is orthogonal then the relations (37) between the raising and the lowering coefficients in equation (27) are easily derived using the 'unitary' condition (10). It is important to emphasize that there are vectors ('patterns') in the action (27) which do not obey the conditions (22). Thus it must be ensured that the proper coefficients for those non-admissible vectors will be zero (the selection rules). For example, when the quantum numbers $\sigma_i$ are changed, then, from the conditions (22), the corresponding weights $h_i$ must be changed according to the following rules:

$$\begin{aligned} \sigma_i \to \sigma_i + 1 \Rightarrow h_i &= -(\sigma_i + 1) \quad -(\sigma_i - 1) \quad -(\sigma_i - 3), \ldots, \sigma_i - 3 \quad \sigma_i - 1 \quad \sigma_i + 1 \\ \sigma_i \to \sigma_i - 1 \Rightarrow h_i &= -(\sigma_i - 1) \quad -(\sigma_i - 3), \ldots, \sigma_i - 3 \quad \sigma_i - 1. \end{aligned} \tag{48}$$

This means that when $h_i = \pm\sigma_i$, then some non-admissible patterns may be possible. Indeed, by the rules (48), when $h_2 = -\sigma_2$ it is possible to have $h_2 = -(\sigma_2 + 1)$ which is not allowed in (48). Consequently, the coefficients $B^+$ and $D^+$ in (27) must be proportional to $(\sigma_2 + h_2)$.

All possible cases are:

$$\begin{aligned}
&E_1^+|\sigma_1,\sigma_2,h_1,-\sigma_2\rangle \to B^+|\sigma_1+1,\sigma_2-1,h_1+1,-(\sigma_2+1)\rangle \Rightarrow B^+ \sim (\sigma_2+h_2)\\
&E_1^+|\sigma_1,\sigma_2,\sigma_1,h_2\rangle \to C^+|\sigma_1-1,\sigma_2+1,\sigma_1+1,h_2-1\rangle \Rightarrow C^+ \sim (\sigma_1-h_1)\\
&E_1^+|\sigma_1,\sigma_2,\sigma_1,h_2\rangle \to D^+|\sigma_1-1,\sigma_2-1,\sigma_1+1,h_2-1\rangle \Rightarrow D^+ \sim (\sigma_1-h_1)\\
&E_1^+|\sigma_1,\sigma_2,h_1,-\sigma_2\rangle \to D^+|\sigma_1-1,\sigma_2-1,h_1+1,-(\sigma_2+1)\rangle \Rightarrow D^+ \sim (\sigma_2+h_2)
\end{aligned} \tag{49}$$

and

$$\begin{aligned}
&E_1^-|\sigma_1,\sigma_2,h_1,\sigma_2\rangle \to B^-|\sigma_1+1,\sigma_2-1,h_1-1,\sigma_2+1\rangle \Rightarrow B^- \sim (\sigma_2-h_2)\\
&E_1^-|\sigma_1,\sigma_2,-\sigma_1,h_2\rangle \to C^-|\sigma_1-1,\sigma_2+1,-(\sigma_1+1),h_2+1\rangle \Rightarrow C^- \sim (\sigma_1+h_1)\\
&E_1^-|\sigma_1,\sigma_2,-\sigma_1,h_2\rangle \to D^-|\sigma_1-1,\sigma_2-1,-(\sigma_1+1),h_2+1\rangle \Rightarrow D^- \sim (\sigma_1+h_1)\\
&E_1^-|\sigma_1,\sigma_2,h_1,\sigma_2\rangle \to D^-|\sigma_1-1,\sigma_2-1,h_1-1,\sigma_2+1\rangle \Rightarrow D^- \sim (\sigma_2-h_2).
\end{aligned} \tag{50}$$

Another set of selection rules is obtained by picking up the extreme values for the quantum numbers $\gamma_{12}$ and $\sigma_1 = \omega_{11}$ in the branching rules (22). For example, when $\sigma_1 = \gamma_{12}$, according to the action (27) the condition $\{\sigma_1 \to \sigma_1+1, \sigma_2 \to \sigma_2+1\}$ corresponds to $\{\gamma_{12} \to \gamma_{12}, \sigma_1 \to \gamma_{12}+1\}$, which would give a non-admissible vector with $\sigma_1 > \gamma_{12}$. In order to avoid this, the coefficient $A^\pm$ must be proportional to $(\gamma_{12}-\sigma_1)$. All other possible cases can be found in a similar way:

$$\begin{aligned}
&A^\pm \sim (\gamma_{12}-\sigma_1) = \tfrac{1}{2}[(\omega_{12}+\omega_{22})-(\sigma_1+\sigma_2)]\\
&B^\pm \sim (\omega_{12}-\gamma_{12}) = \tfrac{1}{2}[(\omega_{12}-\omega_{22})-(\sigma_1-\sigma_2)]\\
&C^\pm \sim (\gamma_{12}-\omega_{22}) = \tfrac{1}{2}[(\omega_{12}-\omega_{22})+(\sigma_1-\sigma_2)]\\
&D^\pm \sim (\sigma_1-(\gamma_{12}-\omega_{22})) = \tfrac{1}{2}[(\sigma_1+\sigma_2)-(\omega_{12}-\omega_{22})].
\end{aligned} \tag{51}$$

Now, using the relations (37), all the selection rules (49)–(51) can be rewritten in a more suitable form,

$$\begin{aligned}
&A^+ = a\{(\sigma_1+h_1+2)(\sigma_2-h_2+2)(\omega_{12}+\omega_{22}-\sigma_1-\sigma_2)(\sigma_1+\sigma_2+2-\omega_{12}+\omega_{22})\}^{\frac{1}{2}}\\
&B^+ = b\{(\sigma_1+h_1+2)(\sigma_2+h_2)(\omega_{12}-\omega_{22}-\sigma_1+\sigma_2)(\omega_{12}-\omega_{22}+\sigma_1-\sigma_2+2)\}^{\frac{1}{2}}\\
&-C^+ = c\{(\sigma_1-h_1)(\sigma_2-h_2+2)(\omega_{12}-\omega_{22}+\sigma_1-\sigma_2)(\omega_{12}-\omega_{22}-\sigma_1+\sigma_2+2)\}^{\frac{1}{2}}\\
&D^+ = d\{(\sigma_1-h_1)(\sigma_2+h_2)(\sigma_1+\sigma_2-\omega_{12}+\omega_{22})(\omega_{12}+\omega_{22}-\sigma_1-\sigma_2+2)\}^{\frac{1}{2}}
\end{aligned} \tag{52}$$

where the normalizing functions $a$–$d$ do not depend on the weights $h_i$. The negative sign in front of $C^+$ was chosen for convenience and corresponds to one of four possibilities for the relative phase choices shown in table 2 (see the next section).

### 3.3. *The recurrence relations*

The normalizing coefficients $a$–$d$ are determined letting the Cartan–Weyl commutation relation

$$[E_1^+, E_1^-] = E_1^+E_1^- - E_1^-E_1^+ = H_1 - H_2$$

act on a generic vector given in (22). Only one of the nine new vectors created on the left-hand side has a counterpart on the right-hand side: the diagonal vector $|\sigma_1,\sigma_2,h_1,h_2\rangle$. The corresponding coefficients to this vector can be written as

$$\begin{aligned}
(h_1-h_2) &+ A^+(\sigma_1,\sigma_2,h_1,h_2)^2 + B^+(\sigma_1,\sigma_2,h_1,h_2)^2 + C^+(\sigma_1,\sigma_2,h_1,h_2)^2\\
&+D^+(\sigma_1,\sigma_2,h_1,h_2)^2 = A^+(\sigma_1-1,\sigma_2-1,h_1-1,h_2+1)^2\\
&+B^+(\sigma_1-1,\sigma_2+1,h_1-1,h_2+1)^2 + C^+(\sigma_1+1,\sigma_2-1,h_1-1,h_2+1)^2\\
&+D^+(\sigma_1+1,\sigma_2+1,h_1-1,h_2+1)^2.
\end{aligned} \tag{53}$$

The new vectors $|\sigma_1 \pm 2, \sigma_2 \pm 2, h_1, h_2\rangle$ and $|\sigma_1 \pm 2, \sigma_2 \mp 2, h_1, h_2\rangle$ give the relations

$$A^+(\sigma_1, \sigma_2, h_1, h_2)D^+(\sigma_1 + 2, \sigma_2 + 2, h_1, h_2) \\ = A^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1)D^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \tag{54}$$

and

$$B^+(\sigma_1, \sigma_2, h_1, h_2)C^+(\sigma_1 + 2, \sigma_2 - 2, h_1, h_2) \\ = B^+(\sigma_1 + 1, \sigma_2 - 1, h_1 - 1, h_2 + 1)C^+(\sigma_1 + 1, \sigma_2 - 1, h_1 - 1, h_2 + 1) \tag{55}$$

respectively. Using (52), the last two relations can be simplified to

$$\begin{aligned} a(\sigma_1, \sigma_2)d(\sigma_1 + 2, \sigma_2 + 2) &= a(\sigma_1 + 1, \sigma_2 + 1)d(\sigma_1 + 1, \sigma_2 + 1) \\ b(\sigma_1, \sigma_2)c(\sigma_1 + 2, \sigma_2 - 2) &= b(\sigma_1 + 1, \sigma_2 - 1)c(\sigma_1 + 1, \sigma_2 - 1) \end{aligned} \tag{56}$$

respectively, whose solutions are

$$\begin{aligned} d(\sigma_1, \sigma_2) &= a(\sigma_1 - 1, \sigma_2 - 1) \\ c(\sigma_1, \sigma_2) &= b(\sigma_1 - 1, \sigma_2 + 1). \end{aligned} \tag{57}$$

The remaining relations

$$A^+(\sigma_1, \sigma_2, h_1, h_2)C^+(\sigma_1 + 2, \sigma_2, h_1, h_2) + B^+(\sigma_1, \sigma_2, h_1, h_2)D^+(\sigma_1 + 2, \sigma_2, h_1, h_2) \\ = A^+(\sigma_1 + 1, \sigma_2 - 1, h_1 - 1, h_2 + 1)C^+(\sigma_1 + 1, \sigma_2 - 1, h_1 - 1, h_2 + 1) \\ + B^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1)D^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \tag{58}$$

and

$$A^+(\sigma_1, \sigma_2, h_1, h_2)B^+(\sigma_1, \sigma_2 + 2, h_1, h_2) + C^+(\sigma_1, \sigma_2, h_1, h_2)D^+(\sigma_1, \sigma_2 + 2, h_1, h_2) \\ = A^+(\sigma_1 - 1, \sigma_2 + 1, h_1 - 1, h_2 + 1)B^+(\sigma_1 - 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \\ + C^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1)D^+(\sigma_1 + 1, \sigma_2 + 1, h_1 - 1, h_2 + 1) \tag{59}$$

come from the new vectors $|\sigma_1 \pm 2, \sigma_2, h_1, h_2\rangle$ and $|\sigma_1, \sigma_2 \pm 2, h_1, h_2\rangle$. These last two relations fix the relative phase to the coefficients $A$–$D$ as shown in table 2.

### *3.4. Solution of the recurrence relations*

Using (52) and (57) the homogeneous relation (53) can be rewritten as

$$\Phi(\sigma_1, \sigma_2)h_2 + \Xi(\sigma_1, \sigma_2)(h_1 - h_2) = 0 \tag{60}$$

where

$$\begin{aligned} \Phi(\sigma_1, \sigma_2) = {} & (\sigma_1 - \sigma_2)(\omega_{12} + \omega_{22} - \sigma_1 - \sigma_2)(\omega_{12} - \omega_{22} - \sigma_1 - \sigma_2 - 2)a^2(\sigma_1, \sigma_2) \\ & - (\sigma_1 - \sigma_2)(\omega_{12} + \omega_{22} - \sigma_1 - \sigma_2 + 2)(\omega_{12} - \omega_{22} - \sigma_1 - \sigma_2)a^2(\sigma_1 - 1, \sigma_2 - 1) \\ & + (\sigma_1 + \sigma_2 + 2)(\omega_{12} - \omega_{22} + \sigma_1 - \sigma_2 + 2)(\omega_{12} - \omega_{22} - \sigma_1 + \sigma_2)b^2(\sigma_1, \sigma_2) \\ & - (\sigma_1 + \sigma_2 + 2)(\omega_{12} - \omega_{22} + \sigma_1 - \sigma_2)(\omega_{12} - \omega_{22} - \sigma_1 + \sigma_2 + 2)b^2(\sigma_1 - 1, \sigma_2 + 1) \end{aligned} \tag{61}$$

and

$$\begin{aligned} \Xi(\sigma_1, \sigma_2) = {} & 1 - 2(\sigma_2 + 2)(\omega_{12} + \omega_{22} - \sigma_1 - \sigma_2)(\omega_{12} - \omega_{22} - \sigma_1 - \sigma_2 - 2)a^2(\sigma_1, \sigma_2) \\ & + 2(\sigma_2)(\omega_{12} + \omega_{22} - \sigma_1 - \sigma_2 + 2)(\omega_{12} - \omega_{22} - \sigma_1 - \sigma_2)a^2(\sigma_1 - 1, \sigma_2 - 1) \\ & + 2(\sigma_2)(\omega_{12} - \omega_{22} + \sigma_1 - \sigma_2 + 2)(\omega_{12} - \omega_{22} - \sigma_1 + \sigma_2)b^2(\sigma_1, \sigma_2) \\ & - 2(\sigma_2 + 2)(\omega_{12} - \omega_{22} + \sigma_1 - \sigma_2)(\omega_{12} - \omega_{22} - \sigma_1 + \sigma_2 + 2)b^2(\sigma_1 - 1, \sigma_2 + 1). \end{aligned} \tag{62}$$

Since, in general, the quantum numbers $h_1$ and $h_2$ are independents, then, from equation (60), the nonlinear system of recurrence equations $\{\Phi = 0, \Xi = 0\}$ must be solved. This is better accomplished by rewriting $\sigma_2 = \omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12}$ where, from (22),

$$\begin{aligned} \gamma_{12} &= \omega_{22} \qquad \omega_{22}+1, \ldots, \omega_{12} \\ \sigma_1 &= \gamma_{12} - \omega_{22} \qquad \gamma_{12} - \omega_{22} + 1, \ldots, \gamma_{12}. \end{aligned} \tag{63}$$

In this way, the previous equations for $\Phi$ and $\Xi$ can be written as

$$\begin{aligned} \Phi(\gamma_{12}, \sigma_1) = {} & 4(2\gamma_{12} - \omega_{12} - \omega_{22})(\gamma_{12} - \sigma_1)(\gamma_{12} - \sigma_1 - 1 - \omega_{22})a^2(\gamma_{12}, \sigma_1) \\ & -4(2\gamma_{12} - \omega_{12} - \omega_{22})(\gamma_{12} - \sigma_1 + 1)(\gamma_{12} - \sigma_1 - \omega_{22})a^2(\gamma_{12}, \sigma_1 - 1) \\ & +4(\omega_{12} + \omega_{22} - 2\gamma_{12} + 2\sigma_1 + 2)(\omega_{12} - \gamma_{12})(\gamma_{12} + 1 - \omega_{22})b^2(\gamma_{12}, \sigma_1) \\ & -4(\omega_{12} + \omega_{22} - 2\gamma_{12} + 2\sigma_1 + 2)(\omega_{12} - \gamma_{12} + 1)(\gamma_{12} - \omega_{22})b^2 \\ & \times(\gamma_{12} - 1, \sigma_1 - 1) \end{aligned} \tag{64}$$

and

$$\begin{aligned} \Xi(\gamma_{12}, \sigma_1) = {} & 1 - 8(\omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12} + 2)(\gamma_{12} - \sigma_1)(\gamma_{12} - \sigma_1 - 1 - \omega_{22})a^2(\gamma_{12}, \sigma_1) \\ & +8(\omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12})(\gamma_{12} - \sigma_1 + 1)(\gamma_{12} - \sigma_1 - \omega_{22})a^2(\gamma_{12}, \sigma_1 - 1) \\ & +8(\omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12})(\omega_{12} - \gamma_{12})(\gamma_{12} + 1 - \omega_{22})b^2(\gamma_{12}, \sigma_1) \\ & -8(\omega_{12} + \omega_{22} + \sigma_1 - 2\gamma_{12} + 2)(\omega_{12} - \gamma_{12} + 1)(\gamma_{12} - \omega_{22})b^2(\gamma_{12} - 1, \sigma_1 - 1). \end{aligned} \tag{65}$$

The solutions are

$$\begin{aligned} a^2(\sigma_1, \sigma_2) &= \frac{1}{64}\frac{(\omega_{12} + \omega_{22} + \sigma_1 + \sigma_2 + 6)(\omega_{12} - \omega_{22} + \sigma_1 + \sigma_2 + 4)}{(\sigma_1 + 1)(\sigma_1 + 2)(\sigma_2 + 1)(\sigma_2 + 2)} \\ b^2(\sigma_1, \sigma_2) &= \frac{1}{64}\frac{(\omega_{12} + \omega_{22} - \sigma_1 + \sigma_2 + 2)(\omega_{12} + \omega_{22} + \sigma_1 - \sigma_2 + 4)}{(\sigma_1 + 1)(\sigma_1 + 2)\sigma_2(\sigma_2 + 1)}. \end{aligned} \tag{66}$$

As expected, no further selection rules are present in the normalizing coefficients $a$ and $b$. As we can see from (22), the right-hand side of (66) has only positive non-null terms.

The procedure used to find the solutions (66) was the most natural one: (I) starting with the lowest value of $\gamma_{12}$, $\gamma_{12} = \omega_{22}$, then a couple of relations begining with the highest value of $\sigma_1$, $\sigma_1 = \gamma_{12}$, were solved for $a$ and $b$ and; (II) using these particular solutions with fixed $\gamma_{12}$, a general form in terms of $\sigma_1$ was found to $a$ and $b$. Steps (I) and (II) were repeated until a general form in terms of $\gamma_{12}$ was also found, too.

## 4. Conclusions

The commutation relations for the generators of the $sp(4)$ algebra were explicitly solved based on the branching rules of the chain $sp(4) \supset sp(2) \oplus sp(2)$. This means that there is a closed formula for calculating any matrix element for all elements of the $sp(4)$ algebra in an arbitrary irreducible representation. All irreducible matrices are real and they satisfy the commutation relations in the Cartan–Weyl form. Therefore, the results presented here are very convenient for applications in which the algebraic handling of several different irreducible representations can be used.

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## Appendix. Matrix elements for the symplectic $sp$(2) algebra

The following is a simple derivation for the matrix elements (25), (26). The commutation relations for the $sp(2)$ algebra in the Cartan–Weyl form can be written as

$$[H_1, E_1^{\pm}] = \pm E_1^{\pm}$$
$$[E_1^{\pm}, E_1^{\mp}] = \pm 2H_1. \tag{67}$$

As usual, the element $E_1^-$ associated with the negative root is obtained from the element $E_1^+$ by a real transposition:

$$E_1^- = (E_1^+)^{\mathrm{T}}. \tag{68}$$

This is all that is needed to write the corresponding unitary irreducible matrices as

$$E_1^+ + E_1^- \qquad \mathrm{i}(E_1^+ - E_1^-). \tag{69}$$

Each unitary irreducible representation for the $sp(2)$ algebra is given by a positive integer $\omega_{11}$ (the highest weight). The $\omega_{11} + 1$ basis vectors are labelled by [20]

$$|\omega_{11}, h_1\rangle \qquad h_1 = \omega_{11}, \omega_{11} - 2, \ldots, -\omega_{11} \tag{70}$$

where the $h_1$ numbers are the eigenvalues of the diagonal element $H_1$,

$$H_1|\omega_{11}, h_1\rangle = h_1|\omega_{11}, h_1\rangle. \tag{71}$$

Using the first relation in (67) and (71), we can see that the $E_1^{\pm}$ elements are raising (lowering) operators with step two:

$$H_1(E_1^{\pm}|\omega_{11}, h_1\rangle) = (h_1 \pm 2)(E_1^{\pm}|\omega_{11}, h_1\rangle).$$

Thus, from the last relation,

$$E_1^{\pm}|\omega_{11}, h_1\rangle = A^{\pm}(\omega_{11}, h_1)|\omega_{11}, h_1 \pm 2\rangle \qquad h_1 = -\omega_{11}, -\omega_{11} + 2, \ldots, \omega_{11}. \tag{72}$$

In order to avoid the non-allowed vectors $|\omega_{11}, \omega_{11} \pm 2\rangle$ when $h_1 = \pm\omega_{11}$, we must have

$$A^{\pm}(\omega_{11}, \pm\omega_{11}) = 0$$

which give the following selection rules:

$$A^{\pm} \sim (\omega_{11} \mp h_1). \tag{73}$$

The coefficients $A^{\pm}$ are related to each other by the condition (68):

$$\begin{aligned}\langle\omega_{11}, h_1|E_1^+|\omega_{11}, h_1 - 2\rangle &= A^-(\omega_{11}, h_1)\langle\omega_{11}, h_1 - 2|\omega_{11}, h_1 - 2\rangle \\ &= A^+(\omega_{11}, h_1 - 2)\langle\omega_{11}, h_1|\omega_{11}, h_1\rangle.\end{aligned}$$

Assuming the basis (70) is orthogonal and normalized, the following relation results from the last equation:

$$A^+(\omega_{11}, h_1) = A^-(\omega_{11}, h_1 + 2). \tag{74}$$

Now, using (73) and (74), we can write the coefficients $A^{\pm}$ in the following form:

$$A^{\pm} = a(\omega_{11})\{(\omega_{11} \mp h_1)(\omega_{11} \pm h_1 + 2)\}^{\frac{1}{2}} \tag{75}$$

where $a(\omega_{11})$ is the normalizing coefficient. Using (72) and (74), the second commutation relation in (67) gives a recurrence relation for $A^{\pm}$:

$$A^-(\omega_{11}, h_1 + 2)^2 = A^-(\omega_{11}, h_1)^2 - 2h_1 \qquad h_1 = -\omega_{11} \qquad -\omega_{11} + 2, \ldots, \omega_{11}. \tag{76}$$

The solution,

$$A^-(\omega_{11}, h_1) = \{\tfrac{1}{2}(\omega_{11} - h_1 + 2)(\omega_{11} + h_1)\}^{\frac{1}{2}} \tag{77}$$

can be found in two distinct ways: (I) directly solving the recurrence relation (76) by letting $h_1$ assume a few values or (II) substituting (75) in (76) to determine $a^2 = \frac{1}{2}$. Therefore, the matrix elements for the $sp(2)$ algebra are:

$$H_1|\omega_{11}, h_1\rangle = h_1|\omega_{11}, h_1\rangle$$
$$E_1^{\pm}|\omega_{11}, h_1\rangle = \{\tfrac{1}{2}(\omega_{11} \mp h_1)(\omega_{11} \pm h_1 + 2)\}^{\frac{1}{2}}|\omega_{11}, h_1 \pm 2\rangle. \tag{78}$$

It follows from this action that the Casimir operator

$$J = H_1^2 + E_1^+E_1^- + E_1^-E_1^+ \tag{79}$$

as expected [4], has the eigenvalue

$$J|\omega_{11}, h_1\rangle = \omega_{11}(\omega_{11} + 2)|\omega_{11}, h_1\rangle. \tag{80}$$

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